

# LARGE N-LIMIT FOR RANDOM MATRICES WITH EXTERNAL SOURCE WITH THREE DISTINCT EIGENVALUES

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**ABSTRACT.** In this paper, we analyze the large  $n$ -limit for random matrix with external source with three distinct eigenvalues. And we confine ourselves in the Hermite case and the three distinct eigenvalues are  $-a, 0, a$ . For the case  $a^2 > 3$ , we establish the universal behavior of local eigenvalue correlations in the limit  $n \rightarrow \infty$ , which is known from unitarily invariant random matrix models. Thus, local eigenvalue correlations are expressed in terms of the sine kernel in the bulk and in terms of the Airy kernel at the edge of the spectrum. The result can be obtained by analyzing  $4 \times 4$  Riemann-Hilbert problem via nonlinear steepest descent method.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

We will consider the ensemble of  $n \times n$  Hermitian matrices  $M$  with the density function defined by

$$\frac{1}{Z_n} e^{-n \text{Tr}(V(M) - AM)} dM, \quad (1.1)$$

where  $V(M)$  is a matrix polynomial with respect to  $M$ , The number  $n$  is a large parameter in the ensemble,  $A$  is a  $n \times n$  matrix, and  $Z_n$  is the normalization constant,

$$Z_n = \int e^{-\text{Tr}(V(M) - AM)} dM, \quad (1.2)$$

where the integral is over all  $n \times n$  Hermitian matrices. When  $A = 0$ , the above ensemble is the standard unitary-invariant ensemble in the theory of random matrices [1]. The case when  $A \neq 0$  is called a random matrix model with external source, and has been studied in, for example, [2, 3, 4, 5, 6, 7, 8,

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*Date:* October 2, 2015.

*Key words and phrases.* Riemann-Hilbert problem, Random matrices, Multiple Orthogonal Polynomials, Large N limit.

9, 10]. In a series of paper [2, 3, 4], Bleher and Kuijlaars considered the large  $n$  limit of this model when external source  $A$  has two distinct eigenvalues via nonlinear steepest descent method based on a  $3 \times 3$  Riemann-Hilbert problem.

In this paper, we consider the case external source  $A$  has three distinct eigenvalues, say  $\{a_1, a_2, a_3\}$ . It was shown that when  $A$  has three distinct eigenvalues, the correlation kernel function  $K_n(x, y)$  is linked to a  $4 \times 4$  Riemann-Hilbert problem: finding  $Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{4 \times 4}$  such that

- (i)  $Y$  is analytic on  $\mathbb{C} \setminus \mathbb{R}$ ,
- (ii) for  $x \in \mathbb{R}$ ,

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega_1(x) & \omega_2(x) & \omega_3(x) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.3)$$

where  $Y_+(x)$  and  $Y_-(x)$  denote the limit of  $Y(z)$  as  $z \rightarrow x$  from upper and lower half plane, respectively.

- (iii) as  $z \rightarrow \infty$ , we have

$$Y(z) = \left( \mathbb{I} + O\left(\frac{1}{z}\right) \right) \text{diag}(z^n, z^{-n_1}, z^{-n_2}, z^{-n_3}), \quad (1.4)$$

where  $\mathbb{I}$  denotes the  $4 \times 4$  identity matrix.

We consider correlation kernel

$$K_n(x, y) = \frac{e^{-\frac{1}{2}(V(x)+V(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{na_1 y} & e^{na_2 y} & e^{na_3 y} \end{pmatrix} Y^{-1}(y) Y(x) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (1.5)$$

and assume that the external source  $A$  is a fixed diagonal matrix with  $n_1$  eigenvalues  $a_1 = a$ ,  $n_2$  eigenvalues  $a_2 = 0$  and  $n_3$  eigenvalues  $a_3 = -a$ . It is the aim of this paper to analyze the Riemann-Hilbert problem as  $n \rightarrow \infty$ , by using the method of nonlinear steepest descent of Deift and Zhou [11]. And we focus here on Gaussian case  $V(x) = \frac{1}{2}x^2$ . The main results of this paper are

**Theorem 1.1.** *Let  $V(M) = \frac{1}{2}M^2$ ,  $n_1 = n_3$ ,  $n_2 = t$ ,  $a^2 > 3$ . Then the limiting mean density of eigenvalues*

$$\rho(z) = \lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x), \quad (1.6)$$

*exists, and it is supported by three intervals,  $[-z_3, -z_2]$ ,  $[-z_1, z_1]$  and  $[z_2, z_3]$ .*

**Theorem 1.2.** *For every  $x_0 \in (-z_3, -z_2) \cup (-z_1, z_1) \cup (z_2, z_3)$  and  $u, v \in \mathbb{R}$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n\rho(x_0)} \hat{K}_n \left( x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)} \right) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \quad (1.7)$$

**Theorem 1.3.** *We use the same notation as in Theorem 1.2. For every  $u, v \in \mathbb{R}$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{(\rho_1 n)^{\frac{2}{3}}} \hat{K}_n \left( z_3 + \frac{u}{(\rho_1 n)^{\frac{2}{3}}}, z_3 + \frac{v}{(\rho_1 n)^{\frac{2}{3}}} \right) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u-v}, \quad (1.8)$$

$$\lim_{n \rightarrow \infty} \frac{1}{(\rho_2 n)^{\frac{2}{3}}} \hat{K}_n \left( z_2 + \frac{u}{(\rho_2 n)^{\frac{2}{3}}}, z_2 + \frac{v}{(\rho_2 n)^{\frac{2}{3}}} \right) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u-v}, \quad (1.9)$$

$$\lim_{n \rightarrow \infty} \frac{1}{(\rho_3 n)^{\frac{2}{3}}} \hat{K}_n \left( z_1 + \frac{u}{(\rho_3 n)^{\frac{2}{3}}}, z_1 + \frac{v}{(\rho_3 n)^{\frac{2}{3}}} \right) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u-v}, \quad (1.10)$$

*Similar limits hold near the edge points  $-z_3, -z_2, -z_1$ .*

## 2. DIFFERENTIAL EQUATIONS

In [15], it was shown that the Riemann-Hilbert problem of  $Y$  has a unique solution

$$Y = \begin{pmatrix} P_{n_1, n_2, n_3} & C(P_{n_1, n_2, n_3} \omega_1) & C(P_{n_1, n_2, n_3} \omega_2) & C(P_{n_1, n_2, n_3} \omega_3) \\ c_1 P_{n_1-1, n_2, n_3} & c_1 C(P_{n_1-1, n_2, n_3} \omega_1) & c_1 C(P_{n_1-1, n_2, n_3} \omega_2) & c_1 C(P_{n_1-1, n_2, n_3} \omega_3) \\ c_2 P_{n_1, n_2-1, n_3} & c_2 C(P_{n_1, n_2-1, n_3} \omega_1) & c_2 C(P_{n_1, n_2-1, n_3} \omega_2) & c_2 C(P_{n_1, n_2-1, n_3} \omega_3) \\ c_3 P_{n_1, n_2, n_3-1} & c_3 C(P_{n_1, n_2, n_3-1} \omega_1) & c_3 C(P_{n_1, n_2, n_3-1} \omega_2) & c_3 C(P_{n_1, n_2, n_3-1} \omega_3) \end{pmatrix} \quad (2.1)$$

where  $P_{n_1, n_2, n_3}(x) = x^n + \dots$  is a monic orthogonal polynomial of degree  $n = n_1 + n_2 + n_3$  with constants

$$c_1 = -2\pi i (h_{n_1-1, n_2, n_3}^{(1)})^{-1}, \quad c_2 = -2\pi i (h_{n_1, n_2-1, n_3}^{(2)})^{-1}, \quad c_3 = -2\pi i (h_{n_1, n_2, n_3-1}^{(3)})^{-1} \quad (2.2)$$

and where  $h_{k_1, k_2, k_3}^{(j)} = \int_{-\infty}^{\infty} P_{k_1, k_2, k_3} x^{k_j} \omega_j(x) dx$ ,  $j = 1, 2, 3$  and  $\omega_j(x) = e^{-(V(x) - a_j x)}$ ,  $j = 1, 2, 3$ , and  $Cf$  denotes the Cauchy transform of  $f$ , that is,

$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s)}{s - z} ds. \quad (2.3)$$

Introducing

$$\Psi_{n_1, n_2, n_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{c_1} & 0 & 0 \\ 0 & 0 & \frac{1}{c_2} & 0 \\ 0 & 0 & 0 & \frac{1}{c_3} \end{pmatrix} Y_{n_1, n_2, n_3} \begin{pmatrix} e^{-V(x)} & 0 & 0 & 0 \\ 0 & e^{-a_1 x} & 0 & 0 \\ 0 & 0 & e^{-a_2 x} & 0 \\ 0 & 0 & 0 & e^{-a_3 x} \end{pmatrix}. \quad (2.4)$$

we can obtain

- (i)  $\Psi_{n_1, n_2, n_3}$  is analytic on  $\mathbb{C} \setminus \mathbb{R}$ ,
- (ii) for  $x \in \mathbb{R}$ , we have

$$\Psi_{n_1, n_2, n_3, +}(x) = \Psi_{n_1, n_2, n_3, -}(x) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.5)$$

- (iii) as  $z \rightarrow \infty$ , we also have

$$\Psi(z) = \left( \mathbb{I} + \frac{\Psi_{n_1, n_2, n_3}^{(1)}}{z} + O\left(\frac{1}{z^2}\right) \right) \begin{pmatrix} z^n e^{-V(x)} & 0 & 0 & 0 \\ 0 & \frac{1}{c_1} z^{-n_1} e^{-a_1 x} & 0 & 0 \\ 0 & 0 & \frac{1}{c_2} z^{-n_2} e^{-a_2 x} & 0 \\ 0 & 0 & 0 & \frac{1}{c_3} z^{-n_3} e^{-a_3 x} \end{pmatrix} \quad (2.6)$$

where

$$\Psi_{n_1, n_2, n_3}^{(1)} = \begin{pmatrix} p_{n_1, n_2, n_3} & \frac{h_{n_1, n_2, n_3}^{(1)}}{h_{n_1-1, n_2, n_3}^{(1)}} & \frac{h_{n_1, n_2, n_3}^{(2)}}{h_{n_1, n_2-1, n_3}^{(2)}} & \frac{h_{n_1, n_2, n_3}^{(3)}}{h_{n_1, n_2, n_3-1}^{(3)}} \\ 1 & \frac{q_{n_1-1, n_2, n_3}^{(1)}}{h_{n_1-1, n_2, n_3}^{(1)}} & \frac{h_{n_1-1, n_2, n_3}^{(2)}}{h_{n_1, n_2-1, n_3}^{(2)}} & \frac{h_{n_1-1, n_2, n_3}^{(3)}}{h_{n_1, n_2, n_3-1}^{(3)}} \\ 1 & \frac{h_{n_1, n_2-1, n_3}^{(1)}}{h_{n_1-1, n_2, n_3}^{(1)}} & \frac{q_{n_1, n_2-1, n_3}^{(2)}}{h_{n_1, n_2-1, n_3}^{(2)}} & \frac{h_{n_1, n_2-1, n_3}^{(3)}}{h_{n_1, n_2, n_3-1}^{(3)}} \\ 1 & \frac{h_{n_1, n_2, n_3-1}^{(1)}}{h_{n_1-1, n_2, n_3}^{(1)}} & \frac{h_{n_1, n_2, n_3-1}^{(2)}}{h_{n_1, n_2-1, n_3}^{(2)}} & \frac{q_{n_1, n_2, n_3-1}^{(3)}}{h_{n_1, n_2, n_3-1}^{(3)}} \end{pmatrix} \quad (2.7)$$

here

$$q_{n_1, n_2, n_3}^{(k)} = \int_{-\infty}^{\infty} P_{n_1, n_2, n_3}(x) x^{n_k+1} \omega_k(x) dx, \quad k = 1, 2, 3. \quad (2.8)$$

and  $P_{n_1, n_2, n_3}(z) = z^n + p_{n_1, n_2, n_3} z^{n-1} + \dots$  is a multiple polynomials.

According to the above notations, we have the following property.

**Proposition 2.1.** *We have the differential equation,*

$$\Psi'_{n_1, n_2, n_3}(z) = N \begin{pmatrix} -z & \frac{n_1}{N} & \frac{n_2}{N} & \frac{n_3}{N} \\ -1 & -a & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & a \end{pmatrix} \Psi_{n_1, n_2, n_3}(z). \quad (2.9)$$

where the prime ' denotes the derivative with respect to  $z$ .

*Proof.* Let

$$A_{n_1, n_2, n_3} = \frac{1}{N} \Psi'_{n_1, n_2, n_3} \Psi_{n_1, n_2, n_3} \quad (2.10)$$

From (2.5), we know  $A_{n_1, n_2, n_3}$  has no jump, that is to say  $A_{n_1, n_2, n_3}$  is analytic on the whole complex plane. From (2.6), we can show as  $z \rightarrow \infty$

$$A_{n_1, n_2, n_3} = \left( \mathbb{I} + \frac{\Psi_{n_1, n_2, n_3}^{(1)}}{+} \dots \right) \begin{pmatrix} -z & 0 & 0 & 0 \\ 0 & -a_1 & 0 & 0 \\ 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & -a_4 \end{pmatrix} \left( \mathbb{I} + \frac{\Psi_{n_1, n_2, n_3}^{(1)}}{+} \dots \right)^{-1} + O\left(\frac{1}{z}\right). \quad (2.11)$$

As  $A_{n_1, n_2, n_3}$  is entire function, it implies that

$$A_{n_1, n_2, n_3} = \begin{pmatrix} -z & c_{n_1, n_2, n_3} & d_{n_1, n_2, n_3} & e_{n_1, n_2, n_3} \\ -1 & -a_1 & 0 & 0 \\ -1 & 0 & -a_2 & 0 \\ -1 & 0 & 0 & -a_3 \end{pmatrix}, \quad (2.12)$$

where  $c_{n_1, n_2, n_3}, d_{n_1, n_2, n_3}, e_{n_1, n_2, n_3}$  are needed to determine.

From the differential equation we get

$$\Psi'_{n_1, n_2, n_3}(z) = N A_{n_1, n_2, n_3} \Psi_{n_1, n_2, n_3}(z). \quad (2.13)$$

Let us show how to determine  $c_{n_1, n_2, n_3}, d_{n_1, n_2, n_3}, e_{n_1, n_2, n_3}$ . From the recursion formula of  $\Psi_{n_1, n_2, n_3}(z)$  [18], say  $\Psi_{n_1+1, n_2, n_3} = U_{n_1, n_2, n_3} \Psi_{n_1, n_2, n_3}$ , where

$$U_{n_1, n_2, n_3} = \begin{pmatrix} z - b_{n_1, n_2, n_3} & -c_{n_1, n_2, n_3} & -d_{n_1, n_2, n_3} & -e_{n_1, n_2, n_3} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & f_{n_1, n_2, n_3} & 0 \\ 1 & 0 & 0 & g_{n_1, n_2, n_3} \end{pmatrix} \quad (2.14)$$

we can get

$$\Psi'_{n_1+1, n_2, n_3} = U'_{n_1, n_2, n_3} \Psi_{n_1, n_2, n_3} + U_{n_1, n_2, n_3} \Psi'_{n_1, n_2, n_3},$$

on the other side, we have

$$\Psi'_{n_1+1, n_2, n_3}(z) = N A_{n_1+1, n_2, n_3} \Psi_{n_1+1, n_2, n_3}(z).$$

This implies

$$U'_{n_1, n_2, n_3} + U_{n_1, n_2, n_3} N A_{n_1, n_2, n_3} = N A_{n_1+1, n_2, n_3} U_{n_1, n_2, n_3}.$$

A simpler calculation shows that

$$\begin{aligned} b_{n_1, n_2, n_3} &= a_1, & c_{n_1+1, n_2, n_3} &= c_{n_1, n_2, n_3} + \frac{1}{N}, \\ d_{n_1+1, n_2, n_3} &= d_{n_1, n_2, n_3}, & e_{n_1+1, n_2, n_3} &= e_{n_1, n_2, n_3}, \\ f_{n_1, n_2, n_3} &= a_2 - a_1, & g_{n_1, n_2, n_3} &= a_3 - a_1. \end{aligned} \quad (2.15)$$

Similarly, we also can show the following formulas via analyze the other recursion relations of  $\Psi_{n_1, n_2, n_3}(z)$ ,

$$\begin{aligned} \tilde{b}_{n_1, n_2, n_3} &= a_2, & \tilde{c}_{n_1+1, n_2, n_3} &= \tilde{c}_{n_1, n_2, n_3}, \\ \tilde{d}_{n_1+1, n_2, n_3} &= \tilde{d}_{n_1, n_2, n_3} + \frac{1}{N}, & \tilde{e}_{n_1+1, n_2, n_3} &= \tilde{e}_{n_1, n_2, n_3}, \\ \tilde{f}_{n_1, n_2, n_3} &= a_1 - a_2, & \tilde{g}_{n_1, n_2, n_3} &= a_3 - a_2. \end{aligned} \quad (2.16)$$

$$\begin{aligned} \hat{b}_{n_1, n_2, n_3} &= a_3, & \hat{c}_{n_1+1, n_2, n_3} &= \hat{c}_{n_1, n_2, n_3}, \\ \hat{d}_{n_1+1, n_2, n_3} &= \hat{d}_{n_1, n_2, n_3}, & \hat{e}_{n_1+1, n_2, n_3} &= \hat{e}_{n_1, n_2, n_3} + \frac{1}{N}, \\ \hat{f}_{n_1, n_2, n_3} &= a_1 - a_3, & \hat{g}_{n_1, n_2, n_3} &= a_2 - a_2. \end{aligned} \quad (2.17)$$

By the initial value  $c_{0, n_2, n_3} = d_{n_1, 0, n_3} = e_{n_1, n_2, 0} = 0$ , we can get

$$c_{n_1, n_2, n_3} = \frac{n_1}{N}, \quad d_{n_1, n_2, n_3} = \frac{n_2}{N}, \quad e_{n_1, n_2, n_3} = \frac{n_3}{N}. \quad (2.18)$$

Noticing our assumption  $a_1 = a, a_2 = 0, a_3 = -a$ , we end the proof.  $\square$

We look for a WKB solution of the differential equation (2.9) of the form

$$\Psi_{n_1, n_2, n_3}(z) = W(z)e^{-N\Lambda(z)}, \quad (2.19)$$

where  $\Lambda$  is a diagonal matrix. By substituting this form into (2.9) we obtain the equation,

$$-W\Lambda'W^{-1} = A + \frac{1}{N}W'W^{-1}, \quad (2.20)$$

where  $A$  is the matrix of coefficients in (2.9). By dropping the last term we reduce it to the eigenvalue problem,

$$W\Lambda'W^{-1} = -A. \quad (2.21)$$

The characteristic polynomial is

$$\begin{aligned} \det[\xi\mathbb{I} + A] &= \begin{vmatrix} \xi - z & t_1 & t_2 & t_3 \\ -1 & \xi - a_1 & 0 & 0 \\ -1 & 0 & \xi - a_2 & 0 \\ -1 & 0 & 0 & \xi - a_3 \end{vmatrix} \\ &= \xi^4 - (z + a_1 + a_2 + a_3)\xi^3 \\ &\quad + [(t_1 + t_2 + t_3) + a_1a_2 + a_1a_3 + a_2a_3 + z(a_1 + a_2 + a_3)]\xi^2 \\ &\quad - [(a_1a_2 + a_1a_3 + a_2a_3)z + (t_1 + t_2)a_3 + (t_1 + t_3)a_2 + (t_2 + t_3)a_1 + a_1a_2a_3]\xi \\ &\quad + a_1a_2t_3 + a_1t_2a_3 + t_1a_2a_3 + a_1a_2a_3z = 0 \end{aligned} \quad (2.22)$$

where  $t_1 = \frac{n_1}{N}$ ,  $t_2 = \frac{n_2}{N}$  and  $t_3 = \frac{n_3}{N}$ .

The above equation defines a Riemann surface, which in the case of interest in this paper (where  $n_1 = n_3$ ,  $a_1 = -a_3 = a$ ,  $a_2 = 0$ ) reduces to

$$\xi^4 - z\xi^3 + (1 - a^2)\xi^2 + a^2z\xi - t_2a^2 = 0. \quad (2.23)$$

This defines the Riemann surface that will play a central role in the rest of the paper.

### 3. SPECTRAL CURVE AND RIEMANN SURFACE

The Riemann surface is given by (2.23) or, if we solve for  $z$ ,

$$z = \frac{\xi^4 + (1 - a^2)\xi^2 - t_2a^2}{\xi^3 - a^2\xi}. \quad (3.1)$$

There are four inverse functions to (3.1), which we choose such that as  $z \rightarrow \infty$ ,

$$\begin{aligned}\xi_1(z) &= z - \frac{1}{z} + O\left(\frac{1}{z^3}\right), \\ \xi_2(z) &= a + \frac{t_1}{z} + O\left(\frac{1}{z^2}\right), \\ \xi_3(z) &= \frac{t_2}{z} + O\left(\frac{1}{z^2}\right), \\ \xi_4(z) &= -a + \frac{t_3}{z} + O\left(\frac{1}{z^2}\right).\end{aligned}\tag{3.2}$$

We need to find the sheet structure of the Riemann surface (2.23). The critical points of  $z(\xi)$  satisfy the equation

$$\xi^6 - (1 + 2a^2)\xi^4 + [a^4 + (3t_2 - 1)a^2]\xi^2 - t_2a^4 = 0,\tag{3.3}$$

Let  $y = \xi^2$ , then the above equation becomes

$$y^3 - (1 + 2a^2)y^2 + [a^4 + (3t_2 - 1)a^2]y - t_2a^4 = 0\tag{3.4}$$

which is a cubic equation as  $y$ . The discriminant of this cubic equation is, and we denote  $a^2$  as  $b$  and  $t_2$  as  $t$ ,

$$\Delta = (1-t)b^2\Delta_c = (1-t)b^2[8b^3 - (9t+15)b^2 + (108t^2 - 90t + 6)b + (1-9t)]\tag{3.5}$$

and then we write  $\Delta_c$  as a function of  $t$ ,

$$\Delta_c(t) = 108bt^2 + (-9b^2 - 90b - 9)t + 8b^3 - 15b^2 + 6b + 1\tag{3.6}$$

This is a quadratic function of  $t$ . And the discriminant of this quadratic function is

$$\Delta_q = -3(b-3)(5b+1)^3,\tag{3.7}$$

We can obtain that if  $b > 3$ , then  $\Delta_c > 0$  for all  $t \in (0, 1)$ , then we have  $\Delta > 0$ . That means the cubic equation (3.4) of  $y$  has three distinct real roots. And in here, we can go further beyond this, that is, the three distinct real roots are all positive. We denote the roots by  $y_1, y_2$  and  $y_3$  (without loss of generally, we can set  $y_1 < y_2 < y_3$ ), and then set

$$p = \sqrt{y_1}, \quad q = \sqrt{y_2}, \quad r = \sqrt{y_3}.\tag{3.8}$$



Then the critical points on the  $\xi$ -plane are  $\pm p, \pm q$  and  $\pm r$ . The branch points on the  $z$ -plane are  $\pm z_1, \pm z_2$  and  $\pm z_3$ , where

$$z_1 = z(\xi = p), \quad z_2 = z(\xi = q), \quad z_3 = z(\xi = r), \quad 0 < z_1 < z_2 < z_3. \quad (3.9)$$

We can show that  $\xi_1, \xi_2, \xi_3$  and  $\xi_4$  have analytic extensions to  $\mathbb{C} \setminus ([-z_3, -z_2] \cup [-z_1, z_1] \cup [z_2, z_3])$ ,  $\mathbb{C} \setminus [z_2, z_3]$ ,  $\mathbb{C} \setminus [-z_1, z_1]$  and  $\mathbb{C} \setminus [-z_3, -z_2]$ . On the cut  $[z_2, z_3]$ ,

$$\begin{aligned} \xi_{1+}(x) &= \overline{\xi_{1-}(x)} = \xi_{2-}(x) = \overline{\xi_{2+}(x)}, \quad z_2 \leq x \leq z_3, \\ \text{Im} \xi_{1+} &> 0, \quad z_2 < x < z_3, \end{aligned} \quad (3.10)$$

and  $\xi_3(x), \xi_4(x)$  are real. On the cut  $[-z_1, z_1]$ ,

$$\begin{aligned} \xi_{1+}(x) &= \overline{\xi_{1-}(x)} = \xi_{3-}(x) = \overline{\xi_{3+}(x)}, \quad -z_1 \leq x \leq z_1, \\ \text{Im} \xi_{1+} &> 0, \quad -z_1 < x < z_1, \end{aligned} \quad (3.11)$$

$\xi_2(x), \xi_4(x)$  are real. On the cut  $[-z_3, -z_2]$ ,

$$\begin{aligned} \xi_{1+}(x) &= \overline{\xi_{1-}(x)} = \xi_{4-}(x) = \overline{\xi_{4+}(x)}, \quad -z_3 \leq x \leq -z_2, \\ \text{Im} \xi_{1+} &> 0, \quad -z_3 < x < -z_2, \end{aligned} \quad (3.12)$$

$\xi_2(x), \xi_3(x)$  are real. See the figure 1.

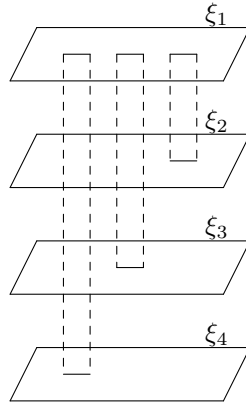


FIGURE 1. Four sheet structure of Riemann surface

We define

$$\rho(x) = \frac{1}{\pi} \text{Im} \xi_{1+}(x), \quad x \in [-z_3, -z_2] \cup [-z_1, z_1] \cup [z_2, z_3], \quad (3.13)$$

then as  $x \in (-z_3, -z_2) \cup (-z_1, z_1) \cup (z_2, z_3)$ , we have  $\rho(x) > 0$  and

$$\begin{aligned} \int_{z_2}^{z_3} \rho(x) dx &= \int_{z_2}^{z_3} \frac{1}{\pi} \text{Im} \xi_{1+}(x) dx \\ &= \int_{z_2}^{z_3} \frac{1}{2\pi i} (\xi_{1+} - \overline{\xi_{1+}})(x) dx \\ &= \int_{z_2}^{z_3} \frac{1}{2\pi i} (\xi_{2-} - \xi_{2+})(x) dx \\ &= \frac{1}{2\pi i} \oint \xi_2(s) ds = \frac{1-t}{2}. \end{aligned} \quad (3.14)$$

Similarly, we have

$$\int_{-z_1}^{z_1} \rho(x) dx = t, \quad \int_{-z_3}^{-z_2} \rho(x) dx = \frac{1-t}{2}. \quad (3.15)$$

And there exists  $\rho_j > 0, j = 1, 2, 3$  such that

$$\begin{aligned} \rho(x) &= \frac{\rho_j}{\pi} |x - z_j|^{\frac{1}{2}} (1 + O(x - z_j)), \quad x \rightarrow z_j, x \in (z_2, z_3), \\ \rho(x) &= \frac{\rho_j}{\pi} |x + z_j|^{\frac{1}{2}} (1 + O(x + z_j)), \quad x \rightarrow -z_j, x \in (-z_3, -z_2), \\ \rho(x) &= \frac{\rho_j}{\pi} |x - z_j|^{\frac{1}{2}} (1 + O(x - z_j)), \quad x \rightarrow z_j, x \in (-z_1, z_1). \end{aligned} \quad (3.16)$$

The reason is there exists a constant  $\rho_3 > 0$  near the branch point  $z_3$  such that as  $z \rightarrow z_3$ ,

$$\begin{aligned} \xi_1(z) &= r + \rho_3(z - z_3)^{\frac{1}{2}} + O(z - z_3), \\ \xi_2(z) &= r - \rho_3(z - z_3)^{\frac{1}{2}} + O(z - z_3). \end{aligned} \quad (3.17)$$

Similarly, there exists a constant  $\rho_2 > 0$  near the branch point  $z_2$  such that as  $z \rightarrow z_2$ ,

$$\begin{aligned} \xi_1(z) &= q - \rho_2(z - z_2)^{\frac{1}{2}} + O(z - z_2), \\ \xi_2(z) &= q + \rho_2(z - z_2)^{\frac{1}{2}} + O(z - z_2). \end{aligned} \quad (3.18)$$

There exists a constant  $\rho_1 > 0$  near the branch point  $z_1$  such that as  $z \rightarrow z_1$ ,

$$\begin{aligned} \xi_1(z) &= p - \rho_1(z - z_1)^{\frac{1}{2}} + O(z - z_1), \\ \xi_2(z) &= p + \rho_1(z - z_1)^{\frac{1}{2}} + O(z - z_1). \end{aligned} \quad (3.19)$$

Next, we need the integrals of the  $\xi$ -function,

$$\lambda_j(z) = \int^z \xi_j(s) ds, \quad j = 1, 2, 3, 4. \quad (3.20)$$

Here, we choose the integral path so that  $\lambda_1(z)$  and  $\lambda_2(z)$  are analytic at  $\mathbb{C} \setminus (-\infty, z_3]$ ,  $\lambda_3(z)$  is analytic at  $\mathbb{C} \setminus (-\infty, z_1]$  and  $\lambda_4(z)$  is analytic at  $\mathbb{C} \setminus (-\infty, -z_2]$ . From (3.2), we know as  $z \rightarrow \infty$

$$\begin{aligned} \lambda_1(z) &= \frac{1}{2} z^2 - \ln z + l_1 + O\left(\frac{1}{z}\right), \\ \lambda_2(z) &= az + \frac{1-t}{2} \ln z + l_2 + O\left(\frac{1}{z}\right), \\ \lambda_3(z) &= t \ln z + l_3 + O\left(\frac{1}{z}\right), \\ \lambda_4(z) &= -az + \frac{1-t}{2} \ln z + l_4 + O\left(\frac{1}{z}\right), \end{aligned} \quad (3.21)$$

where  $l_1, l_2, l_3, l_4$  are constants, which we choose as follows. We choose  $l_1, l_2$  such that

$$\lambda_1(z_3) = \lambda_2(z_3) = 0, \quad (3.22)$$

choose  $l_3$  and  $l_4$  such that

$$\lambda_3(z_1) = \lambda_{1+}(z_1) = \lambda_{1-}(z_1) - (1-t)\pi i, \quad (3.23)$$

$$\lambda_4(-z_2) = \lambda_{1+}(-z_2) = \lambda_{1-}(-z_2) - (1+t)\pi i. \quad (3.24)$$

Then we have the following jump relations:

$$\begin{aligned} \lambda_{1+}(x) &= \lambda_{2-}(x), & \lambda_{1-}(x) &= \lambda_{2+}(x), & x &\in [z_2, z_3], \\ \lambda_{2+}(x) - \lambda_{2-}(x) &= (1-t)\pi i, & & & x &\in (-\infty, z_2], \\ \lambda_{1+}(x) - \lambda_{1-}(x) &= -(1-t)\pi i, & & & x &\in [z_1, z_2], \\ \lambda_{1+}(x) &= \lambda_{3-}(x), & \lambda_{1-}(x) - \lambda_{3+}(x) &= (1-t)\pi i, & x &\in [-z_1, z_1], \\ \lambda_{1+}(x) - \lambda_{1-}(x) &= -(1+t)\pi i, & \lambda_{3+}(x) - \lambda_{3-}(x) &= 2t\pi i, & x &\in [-z_2, -z_1], \\ \lambda_{1+}(x) &= \lambda_{4-}(x), & \lambda_{1-}(x) - \lambda_{4+}(x) &= (1+t)\pi i, & x &\in [-z_3, -z_2], \\ \lambda_{1+}(x) - \lambda_{1-}(x) &= -2\pi i, & \lambda_{4+}(x) - \lambda_{4-}(x) &= (1-t)\pi i, & x &\in (-\infty, -z_3]. \end{aligned} \quad (3.25)$$

For later use, we state the following propositions.

**Lemma 3.1.** (a) *The open interval  $(z_2, z_3)$  has a neighborhood  $U_1$  in the complex plane such that when  $z \in U_1 \setminus (z_2, z_3)$ , the real part  $\operatorname{Re} \lambda_2$  is the biggest one among  $\lambda_j, j = 1, 2, 3, 4$ ;*

(b) *The open interval  $(-z_1, z_1)$  has a neighborhood  $U_2$  in the complex plane such that when  $z \in U_2 \setminus (-z_1, z_1)$ , the real part  $\operatorname{Re} \lambda_3$  is the biggest one among  $\lambda_j, j = 1, 2, 3, 4$ ;*

(c) *The open interval  $(-z_3, -z_2)$  has a neighborhood  $U_3$  in the complex plane such that when  $z \in U_3 \setminus (-z_3, -z_2)$ , the real part  $\operatorname{Re} \lambda_4$  is the biggest one among  $\lambda_j, j = 1, 2, 3, 4$ .*

*Proof.* We just prove (a), the following (b) and (c) are similarly.

First, we have  $\operatorname{Re} \lambda_{3+} < \operatorname{Re} \lambda_{1-}, \operatorname{Re} \lambda_{4+} < \operatorname{Re} \lambda_{1-}$  when  $z \in U_1 \setminus [z_2, z_3]$ . Then, define  $F(x) = \lambda_{2+}(x) - \lambda_{1+}(x)$ , we have  $F(x) = \overline{\lambda_{1+}(x)} - \lambda_{1+}(x)$ . That is to say,  $F(x)$  is pure image on the interval  $(z_2, z_3)$ . As we have  $F'(x) = \xi_{2+}(x) - \xi_{1+}(x) = \xi_{2+}(x) - \xi_{2-}(x) = -2\pi i \rho(x)$ , then  $\operatorname{Im} F'(x) < 0$ . By Cauchy-Riemann equation,  $\operatorname{Re} F(x) > 0$  as we move from the interval  $(z_2, z_3)$  into the upper half-plane.  $\operatorname{Re} F(x) > 0$  as we move from the interval  $(z_2, z_3)$  into the lower half-plane.  $\square$

## 4. TRANSFORMATIONS OF THE RIEMANN-HILBERT PROBLEM

**4.1. First Transform.** Using the functions  $\lambda_j$  and constants  $l_j, j = 1, 2, 3, 4$ , which are defined in the previous section, we define

$$T(z) = \text{diag} \left( e^{-nl_1}, e^{-nl_2}, e^{-nl_3}, e^{-nl_4} \right) Y(z) \text{diag} \left( e^{n(\lambda_1(z) - \frac{1}{2}z^2)}, e^{n(\lambda_2(z) - az)}, e^{n\lambda_3(z)}, e^{n(\lambda_4(z) + az)} \right), \quad (4.1)$$

Then by (1.3) and (4.1), we have  $T_+(x) = T_-(x)J_T(x), x \in \mathbb{R}$ , where

$$J_T(x) = \begin{pmatrix} e^{n(\lambda_{1+} - \lambda_{1-})} & e^{n(\lambda_{2+} - \lambda_{1-})} & e^{n(\lambda_{3+} - \lambda_{1-})} & e^{n(\lambda_{4+} - \lambda_{1-})} \\ 0 & e^{n(\lambda_{2+} - \lambda_{2-})} & 0 & 0 \\ 0 & 0 & e^{n(\lambda_{3+} - \lambda_{3-})} & 0 \\ 0 & 0 & 0 & e^{n(\lambda_{4+} - \lambda_{4-})} \end{pmatrix} \quad (4.2)$$

Be more exact, on  $[z_2, z_3]$

$$J_T(x) = \begin{pmatrix} e^{n(\lambda_{1+} - \lambda_{2+})} & 1 & e^{n(\lambda_{3-} - \lambda_{1-})} & e^{n(\lambda_{4-} - \lambda_{1-})} \\ 0 & e^{n(\lambda_{1-} - \lambda_{2-})} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad (4.3)$$

on  $[-z_1, z_1]$

$$J_T(x) = \begin{pmatrix} e^{n(\lambda_{1+} - \lambda_{3+})} & e^{n(\lambda_{2+} - \lambda_{1-})} & 1 & e^{n(\lambda_{4-} - \lambda_{1-})} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{n(\lambda_{1-} - \lambda_{3-})} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad (4.4)$$

on  $[-z_3, -z_2]$

$$J_T(x) = \begin{pmatrix} e^{n(\lambda_{1+} - \lambda_{1-})} & e^{n(\lambda_{2+} - \lambda_{1-})} & e^{n(\lambda_{3+} - \lambda_{1-})} & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{n(\lambda_{4+} - \lambda_{4-})} \end{pmatrix}; \quad (4.5)$$

on  $\mathbb{R} \setminus ([z_2, z_3] \cup [-z_1, z_1] \cup [-z_3, -z_2])$

$$J_T(x) = \begin{pmatrix} 1 & e^{n(\lambda_{2+} - \lambda_{1-})} & e^{n(\lambda_{3+} - \lambda_{1-})} & e^{n(\lambda_{4+} - \lambda_{1-})} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.6)$$

By (1.4), (3.21) and (4.1), we know the asymptotics of  $T$  is

$$T(z) = \mathbb{I} + O\left(\frac{1}{z}\right), \quad z \rightarrow \infty. \quad (4.7)$$

Thus  $T$  solves the Riemann-Hilbert problem

- $T$  is analytic on  $\mathbb{C} \setminus \mathbb{R}$ ,
- 

$$T_+(x) = T_-(x)J_T(x), \quad x \in \mathbb{R}, \quad (4.8)$$

- As  $z \rightarrow \infty$ ,

$$T(z) = \mathbb{I} + O\left(\frac{1}{z}\right). \quad (4.9)$$

Inserting (4.1) into (1.5), we see that the kernel  $K_n$  can be expressed in terms of  $T$  as follows:

$$K_n(x, y) = \frac{e^{\frac{1}{4}n(x^2 - y^2)}}{2\pi i(x - y)} \begin{pmatrix} 0 & e^{n\lambda_{2+}(y)} & e^{n\lambda_{3+}(y)} & e^{n\lambda_{4+}(y)} \end{pmatrix} T_+^{-1}(y)T_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.10)$$

**4.2. Second Transform.** The second transformation of the Rirmann-Hilbert problem is opening of lenses. Consider a lens with vertices  $z_2, z_3$ , see figure 2. The lens is contained in the neighborhood  $U_1$  of  $(z_2, z_3)$ . We have the

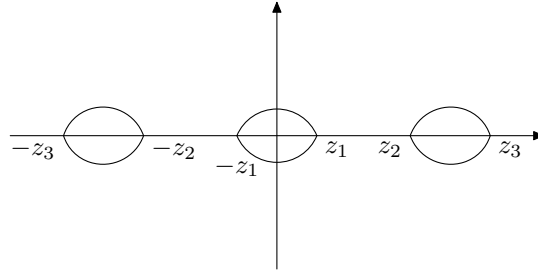


FIGURE 2. The lens of vertices of  $\pm z_j, j = 1, 2, 3$ .

factorization,

$$\begin{aligned}
& \begin{pmatrix} e^{n(\lambda_{1+}-\lambda_{2+})} & 1 & e^{n(\lambda_3-\lambda_{1-})} & e^{n(\lambda_4-\lambda_{1-})} \\ 0 & e^{n(\lambda_{1-}-\lambda_{2-})} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)-} & 1 & -e^{n(\lambda_3-\lambda_2)-} & -e^{n(\lambda_4-\lambda_2)-} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)+} & 1 & e^{n(\lambda_3-\lambda_2)+} & e^{n(\lambda_4-\lambda_2)+} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{4.11}$$

Set

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -e^{n(\lambda_1-\lambda_2)} & 1 & -e^{n(\lambda_3-\lambda_2)} & -e^{n(\lambda_4-\lambda_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ T(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)} & 1 & -e^{n(\lambda_3-\lambda_2)} & -e^{n(\lambda_4-\lambda_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{cases} \tag{4.12}$$

Then (4.22) and (4.12) imply that

$$S_+(x) = S_-(x)J_S(x), \quad J_S(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in [z_2, z_3]. \tag{4.13}$$

Similarly, consider a lens with vertices  $-z_1, z_1$ , that is contained in  $U_2$  and set

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -e^{n(\lambda_1-\lambda_3)} & -e^{n(\lambda_2-\lambda_3)} & 1 & -e^{n(\lambda_4-\lambda_3)} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ T(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ e^{n(\lambda_1-\lambda_3)} & -e^{n(\lambda_2-\lambda_3)} & 1 & -e^{n(\lambda_4-\lambda_3)} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{cases} \quad (4.14)$$

Then (4.22) and (4.14) imply that

$$S_+(x) = S_-(x)J_S(x), \quad J_S(x) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in [-z_1, z_1]. \quad (4.15)$$

Consider a lens with vertices  $-z_3, -z_2$ , that is contained in  $U_3$  and set

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -e^{n(\lambda_1-\lambda_4)} & -e^{n(\lambda_2-\lambda_4)} & -e^{n(\lambda_3-\lambda_4)} & 1 \end{pmatrix} \\ T(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e^{n(\lambda_1-\lambda_4)} & -e^{n(\lambda_2-\lambda_4)} & -e^{n(\lambda_3-\lambda_4)} & 1 \end{pmatrix} \end{cases} \quad (4.16)$$

Then (4.22) and (4.16) imply that

$$S_+(x) = S_-(x)J_S(x), \quad J_S(x) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad x \in [-z_3, -z_2]. \quad (4.17)$$

And set

$$S(z) = T(z) \quad \text{outside of the lens region.} \quad (4.18)$$

Then we have jumps on the boundary of the lenses,

$$S_+(z) = S_-(z)J_S(z) \quad (4.19)$$

where the contours are oriented from left to right (that is, from  $-z_3$  to  $-z_2$ , or from  $-z_1$  to  $z_1$ , or from  $z_2$  to  $z_3$ ). The jump matrix in (4.19) is

$$\begin{aligned} J_S(z) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)} & 1 & e^{n(\lambda_3-\lambda_2)} & e^{n(\lambda_4-\lambda_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ on the upper boundary of the } [z_2, z_3]\text{-lens} \\ J_S(z) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)} & 1 & -e^{n(\lambda_3-\lambda_2)} & -e^{n(\lambda_4-\lambda_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ on the lower boundary of the } [z_2, z_3]\text{-lens} \\ J_S(z) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ e^{n(\lambda_1-\lambda_3)} & e^{n(\lambda_2-\lambda_3)} & 1 & e^{n(\lambda_4-\lambda_3)} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ on the upper boundary of the } [-z_1, z_1]\text{-lens} \\ J_S(z) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ e^{n(\lambda_1-\lambda_3)} & -e^{n(\lambda_2-\lambda_3)} & 1 & -e^{n(\lambda_4-\lambda_3)} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ on the lower boundary of the } [-z_1, z_1]\text{-lens} \\ J_S(z) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e^{n(\lambda_1-\lambda_4)} & e^{n(\lambda_2-\lambda_4)} & e^{n(\lambda_3-\lambda_4)} & 1 \end{pmatrix}, \text{ on the upper boundary of the } [-z_3, -z_2]\text{-lens} \\ J_S(z) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e^{n(\lambda_1-\lambda_4)} & -e^{n(\lambda_2-\lambda_4)} & -e^{n(\lambda_3-\lambda_4)} & 1 \end{pmatrix}, \text{ on the lower boundary of the } [-z_3, -z_2]\text{-lens} \end{aligned} \quad (4.20)$$

And

$$S_+(x) = S_-(x)J_S(x), \quad J_S(x) = J_T(x), \quad x \in (-\infty, -z_3] \cup [-z_2, -z_1] \cup [z_1, z_2] \cup [z_3, \infty). \quad (4.21)$$

Thus  $S$  solves the Riemann-Hilbert problem

- $S$  is analytic on  $\mathbb{C} \setminus (\mathbb{R} \cup \Gamma)$ , where  $\Gamma$  denotes the boundary of the lens,



•

$$S_+(z) = S_-(z)J_S(z), \quad z \in \mathbb{R} \cup \Gamma, \quad (4.22)$$

• As  $z \rightarrow \infty$ ,

$$S(z) = \mathbb{I} + O\left(\frac{1}{z}\right). \quad (4.23)$$

The kernel  $K_n(x, y)$  is expressed in terms of  $S$  as follows. By (4.10) and the definitions (4.12), (4.14), (4.16), for  $x, y$  in  $(z_2, z_3)$ , we have

$$K_n(x, y) = \frac{e^{\frac{n}{4}(x^2 - y^2)}}{2\pi i(x - y)} \begin{pmatrix} -e^{n\lambda_{1+}(y)} & e^{n\lambda_{2+}(y)} & 0 & 0 \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ e^{-n\lambda_{2+}(x)} \\ 0 \\ 0 \end{pmatrix}, \quad (4.24)$$

for  $x, y$  in  $(-z_1, z_1)$ , we have

$$K_n(x, y) = \frac{e^{\frac{n}{4}(x^2 - y^2)}}{2\pi i(x - y)} \begin{pmatrix} -e^{n\lambda_{1+}(y)} & 0 & e^{n\lambda_{3+}(y)} & 0 \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ 0 \\ e^{-n\lambda_{3+}(x)} \\ 0 \end{pmatrix}, \quad (4.25)$$

while for  $x, y$  in  $(-z_3, -z_2)$ , we have

$$K_n(x, y) = \frac{e^{\frac{n}{4}(x^2 - y^2)}}{2\pi i(x - y)} \begin{pmatrix} -e^{n\lambda_{1+}(y)} & 0 & 0 & e^{n\lambda_{4+}(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ 0 \\ 0 \\ e^{-n\lambda_{4+}(x)} \end{pmatrix}. \quad (4.26)$$

Since  $\lambda_{1+}$  and  $\lambda_{2+}$  are complex conjugates on  $(z_2, z_3)$ , we can rewrite (4.24) for  $x, y$  in  $(z_2, z_3)$  as

$$K_n(x, y) = \frac{e^{n(h(y) - h(x))}}{2\pi i(x - y)} \begin{pmatrix} -e^{ni\text{Im}\lambda_{1+}(y)} & e^{-ni\text{Im}\lambda_{1+}(y)} & 0 & 0 \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-ni\text{Im}\lambda_{1+}(x)} \\ e^{ni\text{Im}\lambda_{1+}(x)} \\ 0 \\ 0 \end{pmatrix}, \quad (4.27)$$

where  $h(x) = -\frac{1}{4}x^2 + \operatorname{Re}\lambda_{1+}(x)$ . Similarly, for  $x, y$  in  $(-z_1, z_1)$ , we have

$$K_n(x, y) = \frac{e^{n(h(y)-h(x))}}{2\pi i(x-y)} \begin{pmatrix} -e^{ni\operatorname{Im}\lambda_{1+}(y)} & 0 & e^{-ni\operatorname{Im}\lambda_{1+}(y)} & 0 \end{pmatrix} S_+^{-1}(y)S_+(x) \begin{pmatrix} e^{-ni\operatorname{Im}\lambda_{1+}(x)} \\ 0 \\ e^{ni\operatorname{Im}\lambda_{1+}(x)} \\ 0 \end{pmatrix}, \quad (4.28)$$

for  $x, y$  in  $(-z_3, -z_2)$ , we have

$$K_n(x, y) = \frac{e^{n(h(y)-h(x))}}{2\pi i(x-y)} \begin{pmatrix} -e^{ni\operatorname{Im}\lambda_{1+}(y)} & 0 & 0 & e^{-ni\operatorname{Im}\lambda_{1+}(y)} \end{pmatrix} S_+^{-1}(y)S_+(x) \begin{pmatrix} e^{-ni\operatorname{Im}\lambda_{1+}(x)} \\ 0 \\ 0 \\ e^{ni\operatorname{Im}\lambda_{1+}(x)} \end{pmatrix}. \quad (4.29)$$

**4.3. Model Riemann-Hilbert problem.** As  $n \rightarrow \infty$ , the jump matrix  $J_S(z)$  is exponentially close to the identity matrix at every  $z$  outside of  $[-z_3, -z_2] \cup [-z_1, z_1] \cup [z_2, z_3]$ . This follows from (4.20) and Lemma 3.1 for  $z$  on the boundary of the lenses, and (4.21), (4.6), Lemma 3.1 for  $z$  on the intervals  $(-\infty, -z_3] \cup [-z_2, -z_1] \cup [z_1, z_2] \cup [z_3, \infty)$ .

In this subsection, we solve the following model Riemann-Hilbert problem, where we ignore the exponentially small jumps: find  $M : \mathbb{C} \setminus ([-z_3, -z_2] \cup [-z_1, z_1] \cup [z_2, z_3]) \rightarrow \mathbb{C}^{4 \times 4}$  such that

- $M$  is analytic on  $\mathbb{C} \setminus ([-z_3, -z_2] \cup [-z_1, z_1] \cup [z_2, z_3])$ ,
- 

$$M_+(x) = M_-(x)J_S(x), \quad x \in (-z_3, -z_2) \cup (-z_1, z_1) \cup (z_2, z_3), \quad (4.30)$$

- As  $z \rightarrow \infty$ ,

$$M(z) = \mathbb{I} + O\left(\frac{1}{z}\right) \quad (4.31)$$

This problem is similar to the RH problem considered in [22, Sect. 6.1]. We also follow a similar method to solve it.

We lift the model Riemann-Hilbert problem to the Riemann surface of (2.23) with the sheet structure as in figure 1. Consider

$$\begin{aligned}
\Omega_1 &= \xi_1(\mathbb{C} \setminus ([-z_3, -z_2] \cup [-z_1, z_1] \cup [z_2, z_3])), \\
\Omega_2 &= \xi_2(\mathbb{C} \setminus [z_2, z_3]), \\
\Omega_3 &= \xi_3(\mathbb{C} \setminus [-z_1, z_1]), \\
\Omega_4 &= \xi_4(\mathbb{C} \setminus [-z_3, -z_2]).
\end{aligned} \tag{4.32}$$

Then  $\Omega_1, \Omega_2, \Omega_3, \Omega_4$  give a partition of the complex plane into four regions, see figure 3. The regions  $\Omega_2, \Omega_3, \Omega_4$  are bounded,  $a \in \Omega_2$ ,  $0 \in \Omega_3$ ,  $-a \in \Omega_4$ ,

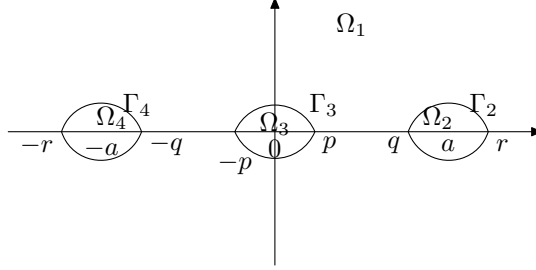


FIGURE 3. Partition of the complex  $\xi$  plane

with the symmetry conditions

$$\bar{\Omega}_2 = \Omega_2, \quad \bar{\Omega}_4 = \Omega_4, \quad \bar{\Omega}_3 = \Omega_3, \quad \Omega_2 = -\Omega_4, \quad \Omega_3 = -\Omega_3. \tag{4.33}$$

Denote by  $\Gamma_k$  the boundary of  $\Omega_k$ ,  $k = 2, 3, 4$ , see figure 3. Then we have

$$\begin{aligned}
\xi_{1+}([z_2, z_3]) &= \xi_{2-}([z_2, z_3]) = \Gamma_2^+ = \Gamma_2 \cap \{\text{Im} z \geq 0\}, \\
\xi_{1-}([z_2, z_3]) &= \xi_{2+}([z_2, z_3]) = \Gamma_2^- = \Gamma_2 \cap \{\text{Im} z \leq 0\}, \\
\xi_{1+}([-z_1, z_1]) &= \xi_{3-}([-z_1, z_1]) = \Gamma_3^+ = \Gamma_3 \cap \{\text{Im} z \geq 0\}, \\
\xi_{1-}([-z_1, z_1]) &= \xi_{3+}([-z_1, z_1]) = \Gamma_3^- = \Gamma_3 \cap \{\text{Im} z \leq 0\}, \\
\xi_{1+}([-z_3, -z_2]) &= \xi_{4-}([-z_3, -z_2]) = \Gamma_4^+ = \Gamma_4 \cap \{\text{Im} z \geq 0\}, \\
\xi_{1-}([-z_3, -z_2]) &= \xi_{4+}([-z_3, -z_2]) = \Gamma_4^- = \Gamma_4 \cap \{\text{Im} z \leq 0\}.
\end{aligned} \tag{4.34}$$

We are looking for a solution  $M$  in the following form:

$$M(z) = \begin{pmatrix} M_1(\xi_1(z)) & M_1(\xi_2(z)) & M_1(\xi_3(z)) & M_1(\xi_4(z)) \\ M_2(\xi_1(z)) & M_2(\xi_2(z)) & M_2(\xi_3(z)) & M_2(\xi_4(z)) \\ M_3(\xi_1(z)) & M_3(\xi_2(z)) & M_3(\xi_3(z)) & M_3(\xi_4(z)) \\ M_4(\xi_1(z)) & M_4(\xi_2(z)) & M_4(\xi_3(z)) & M_4(\xi_4(z)) \end{pmatrix}, \tag{4.35}$$

where  $M_1(\xi), M_2(\xi), M_3(\xi), M_4(\xi)$  are four analytic functions on  $\mathbb{C} \setminus (\Gamma_2 \cup \Gamma_3 \cup \Gamma_4)$ . To satisfy the jump condition (4.30), we need the following relations for  $k = 1, 2, 3, 4$ ,

$$\begin{aligned} M_{k+}(\xi) &= -M_{k-}(\xi), & \xi \in \Gamma_2^+ \cup \Gamma_3^+ \cup \Gamma_4^+, \\ M_{k+}(\xi) &= M_{k-}(\xi), & \xi \in \Gamma_2^- \cup \Gamma_3^- \cup \Gamma_4^-. \end{aligned} \quad (4.36)$$

Since  $\xi_1(\infty) = \infty, \xi_2(\infty) = a, \xi_3(\infty) = 0, \xi_4(\infty) = -a$ , then to satisfy (4.31), we have

$$\begin{aligned} M_1(\infty) &= 1, & M_1(a) &= 0, & M_1(0) &= 0, & M_1(-a) &= 0, \\ M_2(\infty) &= 0, & M_2(a) &= 1, & M_2(0) &= 0, & M_2(-a) &= 0, \\ M_3(\infty) &= 0, & M_3(a) &= 0, & M_3(0) &= 1, & M_4(-a) &= 0, \\ M_4(\infty) &= 0, & M_4(a) &= 0, & M_3(0) &= 0, & M_4(-a) &= 1. \end{aligned} \quad (4.37)$$

Equation (4.36) and (4.37) have the following solution

$$\begin{aligned} M_1(\xi) &= \frac{\xi(\xi^2 - a^2)}{\sqrt{(\xi^2 - p^2)(\xi^2 - q^2)(\xi^2 - r^2)}}, & M_2(\xi) &= c_2 \frac{\xi(\xi + a)}{\sqrt{(\xi^2 - p^2)(\xi^2 - q^2)(\xi^2 - r^2)}}, \\ M_3(\xi) &= c_3 \frac{\xi^2 - a^2}{\sqrt{(\xi^2 - p^2)(\xi^2 - q^2)(\xi^2 - r^2)}}, & M_4(\xi) &= c_4 \frac{\xi(\xi - a)}{\sqrt{(\xi^2 - p^2)(\xi^2 - q^2)(\xi^2 - r^2)}}. \end{aligned} \quad (4.38)$$

Here the constants  $c_2, c_3, c_4$  are defined by  $M_2(a) = 1, M_3(0) = 1, M_4(-a) = 1$ . Notice

$$(\xi^2 - p^2)(\xi^2 - q^2)(\xi^2 - r^2) = \xi^6 - (1 + 2a^2)\xi^4 + [a^4 + (3t - 1)a^2]\xi^2 - ta^4, \quad (4.39)$$

hence

$$M_2(a) = c_2 \frac{2a^2}{\sqrt{2(t - 1)a^4}}. \quad (4.40)$$

By taking into account the cuts of  $M_2(\xi)$ , we have

$$M_2(a) = c_2 i \sqrt{\frac{2}{1 - t}}, \quad (4.41)$$

hence

$$c_2 = -i \sqrt{\frac{1 - t}{2}}. \quad (4.42)$$

Similarly, we have

$$c_3 = -i\sqrt{t}, \quad c_4 = -i\sqrt{\frac{1 - t}{2}}. \quad (4.43)$$

Thus, the solution to the model Riemann-Hilbert is given as following,

$$M(z) = \begin{pmatrix} \frac{\xi_1(\xi_1^2 - a^2)}{\sqrt{(\xi_1^2 - p^2)(\xi_1^2 - q^2)(\xi_1^2 - r^2)}} & \frac{\xi_2(\xi_2^2 - a^2)}{\sqrt{(\xi_2^2 - p^2)(\xi_2^2 - q^2)(\xi_2^2 - r^2)}} & \frac{\xi_3(\xi_3^2 - a^2)}{\sqrt{(\xi_3^2 - p^2)(\xi_3^2 - q^2)(\xi_3^2 - r^2)}} & \frac{\xi_4(\xi_4^2 - a^2)}{\sqrt{(\xi_4^2 - p^2)(\xi_4^2 - q^2)(\xi_4^2 - r^2)}} \\ -i \frac{\sqrt{1-t}\xi_1(\xi_1 + a)}{\sqrt{2(\xi_1^2 - p^2)(\xi_1^2 - q^2)(\xi_1^2 - r^2)}} & -i \frac{\sqrt{1-t}\xi_2(\xi_2 + a)}{\sqrt{2(\xi_2^2 - p^2)(\xi_2^2 - q^2)(\xi_2^2 - r^2)}} & -i \frac{\sqrt{1-t}\xi_3(\xi_3 + a)}{\sqrt{2(\xi_3^2 - p^2)(\xi_3^2 - q^2)(\xi_3^2 - r^2)}} & -i \frac{\sqrt{1-t}\xi_4(\xi_4 + a)}{\sqrt{2(\xi_4^2 - p^2)(\xi_4^2 - q^2)(\xi_4^2 - r^2)}} \\ -i \frac{\sqrt{t}(\xi_1^2 - a^2)}{\sqrt{(\xi_1^2 - p^2)(\xi_1^2 - q^2)(\xi_1^2 - r^2)}} & -i \frac{\sqrt{t}(\xi_2^2 - a^2)}{\sqrt{(\xi_2^2 - p^2)(\xi_2^2 - q^2)(\xi_2^2 - r^2)}} & -i \frac{\sqrt{t}(\xi_3^2 - a^2)}{\sqrt{(\xi_3^2 - p^2)(\xi_3^2 - q^2)(\xi_3^2 - r^2)}} & -i \frac{\sqrt{t}(\xi_4^2 - a^2)}{\sqrt{(\xi_4^2 - p^2)(\xi_4^2 - q^2)(\xi_4^2 - r^2)}} \\ -i \frac{\sqrt{1-t}\xi_1(\xi_1 - a)}{\sqrt{2(\xi_1^2 - p^2)(\xi_1^2 - q^2)(\xi_1^2 - r^2)}} & -i \frac{\sqrt{1-t}\xi_2(\xi_2 - a)}{\sqrt{2(\xi_2^2 - p^2)(\xi_2^2 - q^2)(\xi_2^2 - r^2)}} & -i \frac{\sqrt{1-t}\xi_3(\xi_3 - a)}{\sqrt{2(\xi_3^2 - p^2)(\xi_3^2 - q^2)(\xi_3^2 - r^2)}} & -i \frac{\sqrt{1-t}\xi_4(\xi_4 - a)}{\sqrt{2(\xi_4^2 - p^2)(\xi_4^2 - q^2)(\xi_4^2 - r^2)}} \end{pmatrix} \quad (4.44)$$

The solution  $M$  to the model Riemann-Hilbert problem will be used to construct a parametrix for the Riemann-Hilbert problem for  $S$  outside of a small neighborhood of the edge points. Namely, we will fix some  $r > 0$  and consider the disks of radius  $r$  around the edge points. At the edge points  $M$  is not analytic and in a neighborhood of the edge points the parametrix is constructed differently.

**4.4. Parametrix at Edge Points.** We consider small disks  $D(\pm z_j, r)$ ,  $j = 1, 2, 3$  with radius  $r > 0$  and centered at the edge points, and look for a local parametrix  $P$  defined on the union of the six disks such that

- $P$  is analytic at  $D(\pm z_j, r) \setminus (\mathbb{R} \cup \Gamma)$ ,
- 

$$P_+(z) = P_-(z)J_S(z), \quad z \in (\mathbb{R} \cup \Gamma) \cap D(\pm z_j, r), \quad (4.45)$$

- As  $n \rightarrow \infty$ ,

$$P(z) = \left( \mathbb{I} + O\left(\frac{1}{z}\right) \right) M(z), \quad \text{uniformly for } z \in \partial D(\pm z_j, r). \quad (4.46)$$

We consider here the edge point  $z_3$  in detail. We note that as  $z \rightarrow z_3$ ,

$$\begin{aligned} \lambda_1(z) &= r(z - z_3) + \frac{2\rho_1}{3}(z - z_3)^{\frac{3}{2}} + O(z - z_3)^2, \\ \lambda_2(z) &= r(z - z_3) - \frac{2\rho_1}{3}(z - z_3)^{\frac{3}{2}} + O(z - z_3)^2, \end{aligned} \quad (4.47)$$

thus, we have as  $z \rightarrow z_3$ ,

$$\lambda_1(z) - \lambda_2(z) = \frac{4\rho_1}{3}(z - z_3)^{\frac{3}{2}} + O(z - z_3)^{\frac{5}{2}}. \quad (4.48)$$

Hence

$$\beta(z) = \left[ \frac{3}{4}(\lambda_1(z) - \lambda_2(z)) \right]^{\frac{2}{3}} \quad (4.49)$$

is analytic at  $z_3$ , real-valued on the real axis near  $z_3$ , and  $\beta'(z) = \rho_1^{\frac{2}{3}} > 0$ . So  $\beta$  is a conformal map from  $D(z_3, r)$  to a convex neighborhood of the origin,

if  $r$  is sufficiently small. We take  $\Gamma$  near  $z_3$  such that

$$\beta(\Gamma \cap D(z_3, r)) \subset \{z | \arg(z) = \pm \frac{2\pi}{3}\}. \quad (4.50)$$

Thus,  $\Gamma$  and  $\mathbb{R}$  divide the disk  $D(z_3, r)$  into four regions numbered  $I, II, III$  and  $IV$  such that  $0 < \arg \beta(z) < \frac{2\pi}{3}$ ,  $\frac{2\pi}{3} < \arg \beta(z) < \pi$ ,  $-\pi < \arg \beta(z) < -\frac{2\pi}{3}$  and  $-\frac{2\pi}{3} < \arg \beta(z) < 0$  for  $z$  in regions  $I, II, III$  and  $IV$ , respectively.

Recall that the jumps  $J_S$  near  $z_3$  are given by

$$\begin{aligned} J_S &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in [z_3 - r, z_3), \\ J_S &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & e^{n(\lambda_3 - \lambda_2)} & e^{n(\lambda_4 - \lambda_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ on the upper boundary of the lens in } D(z_3, r) \\ J_S &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & -e^{n(\lambda_3 - \lambda_2)} & -e^{n(\lambda_4 - \lambda_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ on the lower boundary of the lens in } D(z_3, r) \\ J_S &= \begin{pmatrix} 1 & e^{n(\lambda_2 - \lambda_1)} & e^{n(\lambda_3 - \lambda_1)} & e^{n(\lambda_4 - \lambda_1)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in (z_3, z_3 + r]. \end{aligned} \quad (4.51)$$

We write

$$\tilde{P} = \begin{cases} P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -e^{n(\lambda_3 - \lambda_2)} & -e^{n(\lambda_4 - \lambda_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \text{in regions } I, IV, \\ P, & \text{in regions } II, III. \end{cases} \quad (4.52)$$

Then the jumps  $\tilde{P}$  are  $\tilde{P}_+ = \tilde{P}_- J_{\tilde{P}}$ , where

$$\begin{aligned}
 J_{\tilde{P}} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in [z_3 - r, z_3), \\
 J_{\tilde{P}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ on the upper boundary of the lens in } D(z_3, r) \\
 J_{\tilde{P}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ on the lower boundary of the lens in } D(z_3, r) \\
 J_{\tilde{P}} &= \begin{pmatrix} 1 & e^{n(\lambda_2 - \lambda_1)} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in (z_3, z_3 + r].
 \end{aligned} \tag{4.53}$$

The Riemann-Hilbert problem for  $\tilde{P}$  is essentially a  $2 \times 2$  problem, since the jumps (4.53) are non-trivial only in the upper  $2 \times 2$  block. A solution can be constructed in a standard way out of Airy functions. The Airy function  $Ai(z)$  solves the differential equation

$$y'' = zy. \tag{4.54}$$

For any  $\varepsilon > 0$ , in the sector  $\pi + \varepsilon \leq \arg z \leq \pi - \varepsilon$ , it has the asymptotics as  $z \rightarrow \infty$ ,

$$Ai(z) = \frac{1}{2\sqrt{\pi}z^{\frac{1}{4}}} e^{-\frac{2}{3}z^{\frac{3}{2}}} \left(1 + O(z^{-\frac{3}{2}})\right). \tag{4.55}$$

The functions  $Ai(\omega z), Ai(\omega^2 z)$ , where  $\omega = e^{\frac{2\pi i}{3}}$ , also solve the equation (4.54). And we have the linear relation,

$$Ai(z) + \omega Ai(\omega z) + \omega^2 Ai(\omega^2 z) = 0. \tag{4.56}$$

Write

$$y_0(z) = Ai(z), \quad y_1(z) = \omega Ai(\omega z), \quad y_2(z) = \omega^2 Ai(\omega^2 z), \tag{4.57}$$

using these functions to define

$$\phi(z) = \begin{cases} \begin{pmatrix} y_0(z) & -y_2(z) & 0 & 0 \\ y'_0(z) & -y'_2(z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & 0 < \arg z < \frac{2\pi}{3}, \\ \begin{pmatrix} -y_1(z) & -y_2(z) & 0 & 0 \\ -y'_1(z) & -y'_2(z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} -y_2(z) & y_1(z) & 0 & 0 \\ -y'_2(z) & y'_1(z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & -\pi < \arg z < -\frac{2\pi}{3}, \\ \begin{pmatrix} y_0(z) & y_1(z) & 0 & 0 \\ y'_0(z) & y'_1(z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & -\frac{2\pi}{3} < \arg z < 0. \end{cases} \quad (4.58)$$

To match the asymptotic condition as  $z \rightarrow \infty$ , we should have

$$\tilde{P} = \left( \mathbb{I} + O\left(\frac{1}{n}\right) \right) M(z), \quad \text{uniformly in } z \in \partial D(z_3, r). \quad (4.59)$$

Then

$$\tilde{P}(z) = E_n(z) \phi(n^{\frac{2}{3}} \beta(z)) \text{diag} \left( e^{\frac{1}{2}n(\lambda_1(z) - \lambda_2(z))}, e^{-\frac{1}{2}n(\lambda_1(z) - \lambda_2(z))}, 1, 1 \right), \quad (4.60)$$

where

$$E_n = \sqrt{\pi} M \begin{pmatrix} 1 & -1 & 0 & 0 \\ -i & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n^{\frac{1}{6}} \beta^{\frac{1}{4}} & 0 & 0 & 0 \\ 0 & n^{-\frac{1}{6}} \beta^{-\frac{1}{4}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.61)$$

A similar construction works for a parametrix  $P$  around the other edge points.



**4.5. Third Transformation.** Let

$$\begin{aligned} R(z) &= S(z)M(z)^{-1} & z \text{ outside of the disks } D(\pm z_j, r), j = 1, 2, 3, \\ R(z) &= S(z)P(z)^{-1} & z \text{ inside of the disks.} \end{aligned} \quad (4.62)$$

Then we have  $R(z)$  is analytic on  $\mathbb{C} \setminus \Gamma_R$ , where  $\Gamma_R$  consists of the six circles  $\partial D(\pm z_j, r), j = 1, 2, 3$ , the parts of  $\Gamma$  outside of the six disks, and the real intervals  $(-\infty, -z_3 - r), (-z_2 + r, -z_1 - r), (z_1 + r, z_2 - r), (z_3 + r, \infty)$ .

The jump relations for  $R(z)$  are

$$R_+ = R_- J_R, \quad (4.63)$$

where

$$\begin{aligned} J_R &= MP^{-1}, & \text{on the circles, oriented counterclockwise,} \\ J_R &= MJ_S M^{-1}, & \text{on the remaining parts of } \Gamma_R. \end{aligned} \quad (4.64)$$

From (4.46) it follows that  $J_R = \mathbb{I} + O(\frac{1}{n})$  uniformly on the circles, and from (4.20), (4.21), (4.2) and lemma 3.1 it follows that  $J_R = \mathbb{I} + O(e^{-cn})$  for  $c > 0$  as  $n \rightarrow \infty$ , uniformly on the remaining parts of  $\Gamma_R$ . So we can conclude

$$J_R(z) = \mathbb{I} + O(\frac{1}{n}), \quad \text{as } n \rightarrow \infty, \text{ uniformly on } \Gamma_R. \quad (4.65)$$

As  $z \rightarrow \infty$ , we have

$$R(z) = \mathbb{I} + O(\frac{1}{z}). \quad (4.66)$$

From (4.63), (4.65), (4.66) and the fact that we can deform the contours in any desired direction, it follows that

$$R(z) = \mathbb{I} + O(\frac{1}{n|z| + 1}), \quad n \rightarrow \infty, \quad \text{uniformly for } z \in \mathbb{R} \setminus \Gamma_R, \quad (4.67)$$

see [17].

By Cauchy theorem, we then also have

$$R'(z) = O(\frac{1}{n|z| + 1}), \quad (4.68)$$

thus,

$$R^{-1}(y)R(x) = \mathbb{I} + R^{-1}(y)(R(x) - R(y)) = \mathbb{I} + O(\frac{x - y}{n}). \quad (4.69)$$

## 5. PROOFS OF THE RESULTS

**Theorem 5.1.** *Let  $V(M) = \frac{1}{2}M^2$ ,  $n_1 = n_3$ ,  $n_2 = t$ ,  $a^2 > 3$ . Then the limiting mean density of eigenvalues*

$$\rho(z) = \lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x), \quad (5.1)$$

*exists, and it is supported by three intervals,  $[-z_3, -z_2]$ ,  $[-z_1, z_1]$  and  $[z_2, z_3]$ .*

*Proof.* Consider  $x \in (z_2, z_3)$ . We may assume that the circles around the edge points are such that  $x$  is outside of the six disks. From (4.62) it follows that  $S(x) = R(x)M(x)$ . And from (4.69) and  $M_+$  is real analytic in a neighborhood of  $x$ , then

$$S_+^{-1}(y)S_+(x) = \mathbb{I} + O(x - y), \quad y \rightarrow x, \quad (5.2)$$

uniformly for all  $n$ . Thus, we have

$$\begin{aligned} K_n(x, y) &= \frac{e^{n(h(y)-h(x))}}{2\pi i(x-y)} \begin{pmatrix} -e^{ni\text{Im}\lambda_{1+}(y)} & e^{-ni\text{Im}\lambda_{1+}(y)} & 0 & 0 \end{pmatrix} (\mathbb{I} + O(x - y)) \begin{pmatrix} e^{-ni\text{Im}\lambda_{1+}(x)} \\ e^{ni\text{Im}\lambda_{1+}(x)} \\ 0 \\ 0 \end{pmatrix} \\ &= e^{n(h(y)-h(x))} \left( \frac{-e^{ni(\text{Im}\lambda_{1+}(y)-\text{Im}\lambda_{1+}(x))} + e^{-ni(\text{Im}\lambda_{1+}(y)-\text{Im}\lambda_{1+}(x))}}{2\pi i(x-y)} + O(1) \right) \\ &= e^{n(h(y)-h(x))} \left( \frac{\sin(n\text{Im}(\lambda_{1+}(x)-\lambda_{1+}(y)))}{\pi(x-y)} + O(1) \right), \end{aligned} \quad (5.3)$$

Letting  $y \rightarrow x$  and noting that

$$\frac{d}{dy} \text{Im}\lambda_{1+}(y) = \text{Im}\xi_{1+}(y) = \pi\rho(y), \quad (5.4)$$

By L'Hospital rule, we have

$$K_n(x, x) = n\rho(x) + O(1). \quad (5.5)$$

Similarly, for  $x$  in  $(-z_3, -z_2)$ ,  $(-z_1, z_1)$

For  $x \in (-\infty, -z_3) \cup (-z_2, -z_1) \cup (z_1, z_2) \cup (z_3, \infty)$ , we have  $K_n(x, x)$  decreases exponentially fast. This implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) = 0. \quad (5.6)$$

If  $x$  is one of the edge points, it is shown in the proof of 5.3 that as  $n \rightarrow \infty$ ,

$$\frac{1}{n} K_n(x, x) = O\left(\frac{1}{n^{\frac{1}{3}}}\right). \quad (5.7)$$

□

Let

$$\hat{K}_n(x, y) = e^{n(h(x)-h(y))} K_n(x, y), \quad (5.8)$$

where for  $x \in (-z_3, -z_2) \cup (-z_1, z_1) \cup (z_2, z_3)$ ,

$$h(x) = -\frac{1}{4}x^2 + \operatorname{Re}\lambda_{1+}(x), \quad \lambda_{1+}(x) = \int_{z_1}^x \xi_{1+}(s)ds. \quad (5.9)$$

**Theorem 5.2.** *For every  $x_0 \in (-z_3, -z_2) \cup (-z_1, z_1) \cup (z_2, z_3)$  and  $u, v \in \mathbb{R}$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n\rho(x_0)} \hat{K}_n \left( x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)} \right) = \frac{\sin\pi(u-v)}{\pi(u-v)}. \quad (5.10)$$

*Proof.* We just prove  $x \in (z_2, z_3)$ , similarly for the rest. Letting

$$x = x_0 + \frac{u}{n\rho(x_0)}, \quad y = x_0 + \frac{v}{n\rho(x_0)}. \quad (5.11)$$

By the definition of  $\hat{K}_n$ , we have

$$\frac{1}{n\rho(x_0)} \hat{K}_n(x, y) = \frac{\sin(n\operatorname{Im}(\lambda_{1+}(x) - \lambda_{1+}(y)))}{\pi(u-v)} + O\left(\frac{1}{n}\right). \quad (5.12)$$

By the mean value theorem, we have

$$\operatorname{Im}(\lambda_{1+}(x) - \lambda_{1+}(y)) = (x - y)\pi\rho(t), \quad (5.13)$$

for some  $t$  between  $x$  and  $y$ . We also can know that  $t = x_0 + O(1/n)$  and

$$n\operatorname{Im}(\lambda_{1+}(x) - \lambda_{1+}(y)) = \pi(u-v) \frac{\rho(t)}{\rho(x_0)} = \pi(u-v) \left( 1 + O\left(\frac{1}{n}\right) \right). \quad (5.14)$$

Then we have

$$\frac{1}{n\rho(x_0)} \hat{K}_n(x, y) = \frac{\sin\pi(u-v)}{\pi(u-v)} + O\left(\frac{1}{n}\right). \quad (5.15)$$

This is proven 5.2 □

**Theorem 5.3.** *We use the same notation as in Theorem 5.2. For every  $u, v \in \mathbb{R}$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{(\rho_1 n)^{\frac{2}{3}}} \hat{K}_n \left( z_3 + \frac{u}{(\rho_1 n)^{\frac{2}{3}}}, z_3 + \frac{v}{(\rho_1 n)^{\frac{2}{3}}} \right) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u-v}, \quad (5.16)$$

$$\lim_{n \rightarrow \infty} \frac{1}{(\rho_2 n)^{\frac{2}{3}}} \hat{K}_n \left( z_2 + \frac{u}{(\rho_2 n)^{\frac{2}{3}}}, z_2 + \frac{v}{(\rho_2 n)^{\frac{2}{3}}} \right) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u-v}, \quad (5.17)$$

$$\lim_{n \rightarrow \infty} \frac{1}{(\rho_3 n)^{\frac{2}{3}}} \hat{K}_n \left( z_1 + \frac{u}{(\rho_3 n)^{\frac{2}{3}}}, z_1 + \frac{v}{(\rho_3 n)^{\frac{2}{3}}} \right) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u - v}, \quad (5.18)$$

Similar limits hold near the edge points  $-z_3, -z_2, -z_1$ .

*Proof.* We just prove the first formula, similarly for the rest. Noting that  $\beta'(z_3) = \rho_1^{\frac{2}{3}}$ .

Choosing that  $u, v \in \mathbb{R}$  and letting

$$x = z_3 + \frac{u}{(\rho_1 n)^{\frac{2}{3}}}, \quad y = z_3 + \frac{v}{(\rho_1 n)^{\frac{2}{3}}} \quad (5.19)$$

Suppose that  $u, v < 0$ , so we can use the formula (4.27) for  $K_n(x, y)$ . Thus, we have as  $n$  goes to infinity,  $x$  is inside of  $D(z_3, r)$ , from (4.48), (4.60), (4.62), we have

$$\begin{aligned} S_+(x) &= R(x)P_+(x) = R(x)\tilde{P}(x) \\ &= R(x)E_n(x)\Phi_+(n^{\frac{2}{3}}\beta(x))\text{diag} \left( e^{\frac{1}{2}n(\lambda_1(z)-\lambda_2(z))}, e^{-\frac{1}{2}n(\lambda_1(z)-\lambda_2(z))}, 1, 1 \right) \\ &= R(x)E_n(x)\Phi_+(n^{\frac{2}{3}}\beta(x))\text{diag} \left( e^{ni\text{Im}\lambda_{1+}(x)}, e^{-ni\text{Im}\lambda_{1+}(x)}, 1, 1 \right), \end{aligned} \quad (5.20)$$

$S_+(y)$  has similar equation hold. Thus,

$$\begin{aligned} \frac{1}{(\rho_1 n)^{2/3}} \hat{K}_n(x, y) &= \frac{1}{2\pi i(u-v)} \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix} \Phi_+^{-1}(n^{2/3}\beta(y)) E_n^{-1}(y) R^{-1}(y) \\ &\quad \times R(x) E_n(x) \Phi_+(n^{2/3}\beta(x)) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (5.21)$$

Since  $\rho^{2/3} = \beta'(z_3)$ , as  $n \rightarrow \infty$ ,

$$n^{2/3}\beta(x) = n^{2/3}\beta \left( z_3 + \frac{u}{(\rho_1 n)^{2/3}} \right) \rightarrow u, \quad (5.22)$$

Then  $\Phi_+(n^{2/3}\beta(x)) \rightarrow \Phi_+(u)$ . We use the second formula of (4.58) to compute  $\Phi_+(u)$ , we have

$$\lim_{n \rightarrow \infty} \Phi_+(n^{2/3}\beta(x)) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = (-y_1(u) - y_2(u) - y'_1(u) - y'_2(u)00) = \begin{pmatrix} y_0(u) \\ y'_0(u) \\ 0 \\ 0 \end{pmatrix}. \quad (5.23)$$

Similary, we have

$$\lim_{n \rightarrow \infty} \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix} \Phi_+^{-1}(n^{2/3}\beta(y)) = -2\pi i \begin{pmatrix} -y'_0(v) & y_0(v) & 0 & 0 \end{pmatrix}. \quad (5.24)$$

Noting that  $R^{-1}(y)R(x) = \mathbb{I} + O(\frac{x-y}{n})$ , then

$$R^{-1}(y)R(x) = \mathbb{I} + O(\frac{1}{n^{5/3}}). \quad (5.25)$$

From the explicit formula of  $E_n$ ,

$$E_n(x) = O(n^{1/6}), \quad E_n^{-1}(y) = O(n^{1/6}), \quad E_n^{-1}(y)E_n(x) = \mathbb{I} + O(\frac{1}{n^{1/3}}). \quad (5.26)$$

we have

$$\lim_{n \rightarrow \infty} E_n^{-1}(y)R^{-1}(y)R(x)E_n(x) = \mathbb{I}. \quad (5.27)$$

Thus, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{cn^{2/3}} \hat{K}_n(x, y) &= \frac{1}{2\pi i(u-v)} \times (-2\pi i) \begin{pmatrix} -y'_0(v) & y_0(v) & 0 & 0 \end{pmatrix} \begin{pmatrix} y_0(u) \\ y'_0(u) \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{y_0(u)y'_0(v) - y'_0(u)y_0(v)}{u-v}. \end{aligned} \quad (5.28)$$

Since  $y_0 = Ai$ , then we prove the first equation as  $u, v < 0$ . The rest is similarly, we just use the another relation between  $P$  and  $\tilde{P}$ .  $\square$

**Acknowledgements** This work of Xu was supported by Shanghai Sailing Program supported by Science and Technology Commission of Shanghai Municipality under Grant NO. 15YF1408100 and the Hujiang Foundation of China (B14005). Most of this work was done during the author(Xu) visited Indiana University-Purdue University, Indianapolis. And the author also would like to thank Prof. Bleher for his advise and help and arranging this visit. Fan was support by grants from the National Science Foundation of China (Project No. 11271079). Yang thank the Macau Science and Technology Development Fund for generous support: MYRG2014-00011-FST from Macau

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LARGE N-LIMIT FOR RANDOM MATRICES WITH EXTERNAL SOURCE WITH 3 EIGENVALUES

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